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On two-dimensional percolation

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Abstract. We present new series data for both high- and low-density bond and site percolation on the square lattice. The series have been obtained by the finite-lattice method, and in all cases extend pre-existing series. An analysis of these series gives refined estimates of critical points, critical exponents and amplitudes for bond and site animals, and for the percolation probability and mean-size exponents.

1. Introduction and analysis

In [1] we showed how the finite-lattice method (hereinafter abbreviated to FLM) can be used to generate series expansions for percolation problems. In this work we present the longer series obtained from the FLM, and an analysis of the new data.

In table 1 we give the zeroth-, first- and second-site perimeter moment coefficients, enumerated by area. Here, $g_{s,t}$ is the number of connected clusters with site area s and site perimeter t. As usual, p is the probability that a site (or bond) is occupied, and $q = 1 - p$. In table 2 we give the corresponding series enumerated by perimeter.

The second column of table 1 extends by one coefficient the series given some 13 years previously by Redelmeier [2], in 0.2% of the computer time used by Redelmeier's highly optimized program. Analysis of this series by first- and second-order differential approximants confirms and refines an earlier analysis [3] that the coefficients grow asymptotically as

$$As^{-1}\lambda^{s} \qquad \text{where } \lambda = 4.062\,65 \pm 0.000\,05.$$

Analysis of the higher moments reveals that the kth perimeter moment behaves asymptotically as $A_k s^{-1+k}\lambda^{s}$. Assuming the above estimate of λ, we find from the data in table 1 that

$$A_0 = 0.3160 \pm 0.0005 \qquad A_1 = 0.3777 \pm 0.0006 \qquad A_2 = 0.4517 \pm 0.0005.$$

It is significant that the amplitudes are increasing by a constant multiple of 1.195, as this implies that the mean perimeter

$$\langle p\rangle_n \sim 1.195n \qquad \text{and} \qquad \langle p^2\rangle_n \sim 1.195^2 n^2.$$

This implies zero variance, a result which is only true to leading order. An analysis of the variance confirms this, and we find

$$\mathrm{var}(\langle p\rangle_n) = \langle p^2\rangle_n - \langle p\rangle_n^2 \approx 0.3885n.$$

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Table 1. Raw enumeration of site data for enumeration by area (perimeter moments).

n	Coefficient of x^n in $\sum_{s,t} x^s g_{s,t}$	$\sum_{s,t} tx^s g_{s,t}$	$\sum_{s,t} t^2 x^s g_{s,t}$
0	0	0	0
1	1	4	16
2	2	12	72
3	6	44	324
4	19	164	1424
5	63	624	6224
6	216	2412	27140
7	760	9436	118088
8	2725	37205	512089
9	9910	147488	2212964
10	36446	587018	9531954
11	135268	2343620	40933836
12	505861	9379367	175299075
13	1903890	37609788	748816324
14	7204874	151047810	3191269054
15	27394666	607429388	13571511648
16	104592937	2445415156	57602967180
17	400795844	9853980544	244050892096
18	1540820542	39738767634	1032273460510
19	5940738676	160366235260	4359548367412
20	22964779660	647542173314	18385211158762
21	88983512783	2616056891412	77431787430920
22	345532572678	10573603789434	325710977158318
23	1344372335524	42753589936592	1368494742069048
24	5239988770268	172932369469100	5743589522421572
25	20457802016011		

These numerical results can be checked without assuming the value of λ given above by extrapolating the sequence given by the quotient of terms in the third and second column of table 1. This quotient is independent of λ, and is just the sequence $\langle p \rangle_n$. Replacing column 3 by column 4 gives the sequence $\langle p^2 \rangle_n$.

For directed animals [4] we find a similar situation, with

$$\langle p \rangle_n \sim 0.75n \qquad \text{and} \qquad \langle p^2 \rangle_n \sim 0.75^2 n^2.$$

We remark that the first result is exact, the second only numerical. For square and triangular lattice polygons [5, 6], we also find that $\langle p \rangle_n \sim An$. This is essentially one-dimensional behaviour.

Analysis of the perimeter enumeration data in table 2 suggests that the coefficients do not increase in the usual manner, that is, as $Bs^{\alpha}\kappa^{s}$, but somewhat faster. Indeed, it can easily be shown that the number of connected clusters with site area s and site perimeter t enumerated by perimeter increases super-exponentially. This can be seen by considering an animal consisting of a n by n square (perimeter $4n$ so far) with n holes in the interior (perimeter $8n$). In order to obtain a lower bound, we allow only alternate sites to be occupied by a hole. To be more precise, let the individual squares or cells be identified by integer Cartesian coordinates. Then the holes can only occupy those cells with *even* coordinates. This then precludes holes occupying adjacent sites, and hence prevents the animal from becoming disconnected. Thus only a fraction of $\frac{1}{4}$ of the available cells are available for

Table 2. Raw enumeration of site data for enumeration by perimeter (area moments).

n	Coefficient of x^n in		
	$\sum_{s,t} x^t g_{s,t}$	$\sum_{s,t} s x^t g_{s,t}$	$\sum_{s,t} s^2 x^t g_{s,t}$
1	0	0	0
2	0	0	0
3	0	0	0
4	1	1	1
5	0	0	0
6	2	4	8
7	4	12	36
8	12	47	187
9	32	156	772
10	110	658	4010
11	340	2424	17632
12	1209	10090	86118
13	4272	41028	403428
14	16166	176864	1983072
15	61848	762716	9645236
16	246660	3399962	48085206
17	1004883	15366932	241194884
18	4209124	70960296	1228200800
19	18020832	333061552	6320830596
20	78898047	1590691053	32934623685
21	352437205	7717451656	173552604240
22	1605225878	38026124266	925108119430
23	7445515638	190132975588	4986158657876
24	35142033027	964288543526	27170686577418
25	168644213617	4957901690760	149655652437584
26	822311934788	25831248567708	833044055206460
27	4071431204506	136323179341492	4685351988769396
28	20457850555113	728468807117358	26621981398894650

occupation by holes. There will be

$$\binom{\frac{n^2}{4}}{n}$$

of arranging the n holes in the $n^2/4$ available positions. Actually this is a lower bound, as we are restricting the sites that a hole may occupy. This directly gives a super-exponential $\binom{n^2/4}{n} \sim (\frac{n}{4})^n$ number of animals for a linear number of perimeter sites, as required. More rigorous and tighter bounds are straightforward to construct.

However, the quotient of the corresponding coefficients in columns 3 and 2 of table 2 gives a series for the mean area, while the quotient of the coefficients in columns 4 and 2 gives a series for the mean square area. These two series are not as well behaved or as easy to extrapolate as the corresponding perimeter series discussed above. It seems from our analysis that

$$\langle a \rangle_n \sim A n^{1.5} \qquad \text{and that} \qquad \langle a^2 \rangle_n \sim B n^3$$

but the results are not totally convincing. We offer these as the most likely simple, rational exponents. Note that these are also the exponents found for the mean area and mean square area for polygons [7]. Accepting these exponents as exact, we find the following values for the amplitudes:

$$A = 0.30 \pm 0.02 \qquad \text{and} \qquad B = 0.090 \pm 0.009.$$

Again we see that $B \approx A^2$, implying zero variance to leading order. If we accept these exponent values, an analysis of the variance suggests that

$$\operatorname{var}(\langle a \rangle_n) = \langle a^2 \rangle_n - \langle a \rangle_n^2 \approx 0.0163\sqrt{n}$$

but this must be regarded as rather tentative. For square lattice polygons [7] the corresponding amplitude values are $A \approx 0.1416$ and $B \approx 0.0213$, so we see that $B > A^2$ in this case.

Table 3. Raw enumeration of site data for low-density percolation.

n	Coefficient of p^n in $\sum_{s,t} p^s q^t g_{s,t}$	$\sum_{s,t} s p^s q^t g_{s,t}$	$\sum_{s,t} s^2 p^s q^t g_{s,t}$
0	0	0	0
1	1	1	1
2	−2	0	4
3	0	0	12
4	1	0	24
5	0	0	52
6	0	0	108
7	0	0	224
8	1	0	412
9	−1	0	844
10	2	0	1528
11	−4	0	3152
12	11	0	5036
13	−26	0	11984
14	62	0	15040
15	−142	0	46512
16	333	0	34788
17	−780	0	197612
18	1828	0	4036
19	−4256	0	929368
20	9894	0	−702592
21	−23007	0	4847552
22	53682	0	−7033956
23	−125690	0	27903296
24	295221	0	−54403996
25	−694759	0	170579740

Tables 3 and 4 give the first three area moments of site data for low-and high-density percolation respectively, where we use the accepted notation $q = 1 - p$. The last column of table 3 (the second moment) is the low-density mean-size data. A recent analysis of this series (with one fewer coefficient) is given in [8]. Using both traditional Dlog Padé approximants and more powerful methods based on the Roskies [9] transformation (the latter can accommodate non-analytic confluent terms) [8] yields

$$p_c = 0.59275 \pm 0.0001 \qquad \text{and} \qquad \gamma = 2.38 \pm 0.05.$$

We have analysed the one-term longer series by differential approximants (which account for confluent and analytic corrections to scaling) and obtain very similar results, notably

$$p_c = 0.5928 \pm 0.0002 \qquad \text{and} \qquad \gamma = 2.40 \pm 0.04.$$

Table 4. Raw enumeration of site data for high-density percolation.

n	Coefficient of q^n in $\sum_{s,t} p^s q^t g_{s,t}$	$\sum_{s,t} s p^s q^t g_{s,t}$	$\sum_{s,t} s^2 p^s q^t g_{s,t}$
1	0	0	0
2	0	0	0
3	0	0	0
4	1	1	1
5	−1	−1	−1
6	2	4	8
7	0	4	20
8	2	15	87
9	−3	5	125
10	20	158	1266
11	−58	−234	−170
12	163	1349	13353
13	−409	−2713	−12133
14	1318	13704	164364
15	−4400	−42676	−370768
16	14526	172825	2298461
17	−45609	−559053	−6677661
18	142904	2029776	31332020
19	−447914	−6774936	−103144904
20	1416957	23900386	429747186
21	−4493802	−81129962	−1500383110
22	14317184	282099620	5872475248
23	−45743704	−963894132	−21026317880
24	146776574	3331512669	79599546793
25	−472408139	−11422580633	−287835865137
26	1524584800	39350336472	1068454376376
27	−4927578504	−134939821080	−3869679402012
28	15944656731	463383554563	14178781096019
29	−51633916931	−1586767676943	−51239732389715
30	167349948780	5434335886108	186017185734256

Recently Ziff [10] obtained the extremely accurate estimate $p_c = 0.5927460 \pm 0.0000005$ by simulations and extrapolation to infinite systems. Biasing the approximants at this value of p_c allows a somewhat more precise estimate of $\gamma = 2.392 \pm 0.007$ to be made. From conformal invariance theory it is believed that $\gamma = \frac{43}{18} = 2.3888\ldots$, so all the numerical results are consistent with this.

The data in table 4 are the high-density analogue of table 3 data, and so is expanded in powers of $q = 1 - p$. The last column and second last column give the percolation probability and mean-size series. To calculate the percolation probability, $P(p)$, one must divide the third column by p and subtract the quotient from 1. To calculate the mean-size, $S(p)$, one must divide the fourth column (as a polynomial) by the third column. The result of these manipulations is given in table 9.

Tables 5 and 6 are the *bond* analogues of the data presented in tables 1 and 2. Thus table 5 gives perimeter moments for enumeration by area, and table 6 gives area moments for enumeration by perimeter. Analysis of the series in table 5 shows that the coefficients in the series grow asymptotically as $C_k s^{-1+k} \mu^s$, where

$$\mu = 5.2082 \pm 0.0014 \qquad \text{and} \qquad C_0 = 0.5084 \pm 0.0003.$$

Table 5. Raw enumeration of bond data for enumeration by area (perimeter moments).

	Coefficient of x^n in		
n	$\sum_{s,t} x^s g_{s,t}$	$\sum_{s,t} t x^s g_{s,t}$	$\sum_{s,t} t^2 x^s g_{s,t}$
0	0	0	0
1	2	12	72
2	6	48	384
3	22	216	2124
4	88	1020	11856
5	372	4956	66264
6	1628	24482	369646
7	7312	122368	2056732
8	33466	616494	11408174
9	155446	3124292	63086576
10	730534	15903412	347841504
11	3466170	81229492	1912625616
12	16576874	416014902	10489772682
13	79810756	2135241008	57394849480
14	386458826	10978780964	313349139916
15	1880580352	56532459496	1707282827524
16	9190830700	291456857716	9284745918068
17	45088727820	1504184058588	50406017097600
18	221945045488	7769835892168	273210333409000

The mean perimeter $\langle p\rangle_n$ and mean-square perimeter $\langle p^2\rangle_n$ are given by the quotients of columns 3 and 2, and columns 4 and 2, respectively. We find

$$\langle p\rangle_n \sim 1.634n \qquad \text{and} \qquad \langle p^2\rangle_n \sim 2.670n^2.$$

Thus we again find zero variance to leading order, as for the site data. More detailed analysis permits us to estimate

$$\mathrm{var}(\langle p\rangle_n) = \langle p^2\rangle_n - \langle p\rangle_n^2 \approx 0.48n.$$

The quotient of the corresponding coefficients in columns 3 and 2 of table 6 gives a series for the mean area, while the quotient of the coefficients in columns 4 and 2 gives a series for the mean-square area. As in the site case, these two series are not as well behaved, nor as easy to extrapolate, as the corresponding perimeter-moment series discussed above. Again, our analysis suggests that $\langle a\rangle_n \sim An^{1.5}$ and that $\langle a^2\rangle_n \sim Bn^3$, but again the results are not totally convincing. We offer these as the most likely simple, rational exponents, and note that they agree with the site analysis above. Accepting these exponents, we then estimate $A \approx 0.292$ and $B \approx 0.088$, which again are consistent with zero variance to leading order. If we accept these exponent values, an analysis of the variance suggests that

$$\mathrm{var}(\langle a\rangle_n) = \langle a^2\rangle_n - \langle a\rangle_n^2 \approx 0.014\sqrt{n}$$

but, as for the site case, this too must be regarded as rather speculative.

Similarly, tables 7 and 8 are *bond* analogues of the data given in tables 3 and 4. Thus the last column of table 7 gives *twice* the low-density mean-size series. Table 8 provides the raw data necessary to calculate the high-density bond series. Dividing the third column of table 8 by $2p$ and subtracting the result from column 1 gives the percolation probability, while to obtain $S(p)$ one must divide the fourth column (as a polynomial) by the third column. The results of these calculations are given in table 10.

Table 6. Raw enumeration of bond data for enumeration by perimeter (area moments).

	Coefficient of x^n in		
n	$\sum_{s,t} x^t g_{s,t}$	$\sum_{s,t} s x^t g_{s,t}$	$\sum_{s,t} s^2 x^t g_{s,t}$
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	2	2	2
7	0	0	0
8	7	16	40
9	4	12	36
10	28	108	460
11	46	212	1016
12	154	858	5222
13	388	2564	17956
14	1210	9024	72340
15	3390	29564	274708
16	10997	106742	1104446
17	33938	371596	4327464
18	111730	1358978	17560918
19	371546	5000420	71337228
20	1270598	18785454	294158258
21	4423576	71621156	1226011836
22	15763826	277894978	5173273934
23	57172746	1093826984	22068841044
24	211404844	4372425680	95257047916
25	795372138	17729839760	415786114276
26	3044659810	72921469200	1835329934024
27	11846314208	304038606700	8190983994160
28	46831536088	1284748753026	36957042888114
29	187998746668	5500045485084	168549233872572
30	766020610618	23847716140138	776906013726950
31	3166647572118	104696989759720	3618777210942168
32	13275958960735	465281314729004	17031338614913164
33	56425710438434	2092576662295336	80978726301629704
34	243044111103808	9522024048400624	388930070434619124
35	1060598007196010	43829028586609732	1886662705862726136
36	4687529484048236	204026497955080580	9242383210598391084
37	20976931873711572	960318168870200392	45717912049238544924
38	95023034711157024	4569441842243693736	228322224919409302396
39	435607229985549124	21976051608639892452	1151110316189447822944

Analysing the bond mean-size series by differential approximants gives

$$p_c(\text{bond}) = 0.5007 \qquad \text{and} \qquad \gamma = 2.43$$

from first-order differential approximants, and

$$p_c(\text{bond}) = 0.5002 \qquad \text{and} \qquad \gamma = 2.41$$

from second-order differential approximants. Biasing the approximants at the known exact value of $p_c = \frac{1}{2}$ give $\gamma = 2.389 \pm 0.001$, in excellent agreement with the presumed exact value of 2.3888 . . . .

Table 7. Raw enumeration of bond data for low-density percolation.

n	Coefficient of p^n in $\sum_{s,t} p^s q^t g_{s,t}$	$\sum_{s,t} s p^s q^t g_{s,t}$	$\sum_{s,t} s^2 p^s q^t g_{s,t}$
0	0	0	0
1	2	2	2
2	−6	0	12
3	4	0	36
4	0	0	96
5	0	0	252
6	2	0	600
7	−2	0	1524
8	7	0	3336
9	−12	0	8432
10	28	0	17336
11	−54	0	43976
12	115	0	86116
13	−236	0	221664
14	530	0	404864
15	−1238	0	1122040
16	3041	0	1750764
17	−7430	0	5762572
18	17906	0	7002112

In tables 9 and 10 we give the high-density percolation probability $P(p)$ and mean-size series $S(p)$ for both site and bond percolation respectively, expanded in powers of $q = 1-p$. Analysis of the site series by inhomogeneous differential approximants gave only coarse critical probability and exponent estimates. For the percolation probability, unbiased approximants yielded $q_c = 0.42 \pm 0.02$, compared to the best Monte Carlo estimate [10] of $q_c = 0.407\,254\,024$, with an exponent estimate around $\beta = 0$. Analysis of the series by Dlog Padé approximants yields the following unbiased estimates of the critical probability, $q_c = 0.407 \pm 0.001$ with exponent $\beta = 0.136 \pm 0.009$. Using Padé analysis, and biasing the *exponent* at $\frac{5}{36}$ gave $q_c = 0.4073 \pm 0.0001$. Hence $p_c = 0.5927 \pm 0.0001$, which can be compared to the best Monte Carlo estimate [10] of $p_c = 0.592\,7460 \pm 0.000\,0005$. Biasing the approximants with the precise estimate of q_c gave the estimate $\beta = 0.138 \pm 0.001$ from inhomogeneous differential approximants, and $\beta = 0.1383 \pm 0.001$ from Dlog Padé approximants.

Analysis of the mean-size series gave similarly imprecise estimates from unbiased approximants, while biased inhomogeneous differential approximants gave $\gamma' \approx 2.0$, with $\gamma' \approx 1.8$ being the best estimate obtained from biased Dlog Padé approximants.

Analysis of the bond series $P(p)$ by unbiased differential approximants was, as for the site series, not particularly revealing. A critical percolation probability $q_c = 0.5004 \pm 0.0006$ was found, with exponent $\beta = 0.10 \pm 0.08$. However biased Padé approximants, using the known value $q_c = 0.5$, gave $\beta = 0.1387 \pm 0.0003$, in excellent agreement with the presumed exact value, $\beta = \frac{5}{36} = 0.138\,88\ldots$, while biasing the exponent at $\frac{5}{36}$ gave the estimate $q_c = 0.500\,00 \pm 0.000\,08$.

Unbiased differential approximant analysis of the mean-size series yielded

$$q_c = 0.5000 \pm 0.0007 \qquad \text{and} \qquad \gamma' = 2.04 \pm 0.08.$$

Biased differential approximant analysis gave $\gamma' = 2.056 \pm 0.02$, which can be compared

Table 8. Raw enumeration of bond data for high-density percolation.

n	Coefficient of q^n in $\sum_{s,t} p^s q^t g_{s,t}$	$\sum_{s,t} s p^s q^t g_{s,t}$	$\sum_{s,t} s^2 p^s q^t g_{s,t}$
1	0	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	2	2	2
7	−2	−2	−2
8	7	16	40
9	−12	−28	−76
10	28	108	472
11	−54	−228	−1112
12	115	690	4674
13	−236	−1574	−11906
14	530	4406	41730
15	−1238	−10966	−112558
16	3041	30566	364838
17	−7430	−79782	−1022134
18	17906	216432	3144460
19	−42122	−561012	−8864580
20	98603	1478748	26205524
21	−233406	−3838780	−73866120
22	566384	10169288	214494080
23	−1400660	−26895316	−607694848
24	3491205	71772016	1748569328
25	−8689744	−190412848	−4949055280
26	21603828	506071304	14097525720
27	−53837346	−1342438468	−39773663592
28	134798956	3573457212	112666591000
29	−338536144	−9512045212	−317453437728
30	851068458	25342257280	895666137236
31	−2142122208	−67458334432	−2516876996064
32	5410012703	179764504434	7074242270754
33	−13725274670	−479550804278	−19839870974894
34	34939580562	1281123514138	55646364860530
35	−89059616514	−3423303956978	−155813381269622
36	227116742018	9148106476580	436018322499176
37	−579885455698	−24447977561936	−1218405699287324
38	1483978267194	65379524204374	3402881039506162
39	−3807376632980	−174966302968974	−9495529737282334

with the presumed exact value, $\gamma = \frac{43}{18} = 2.3888\ldots$. This estimate, lying well outside the expected exact value, is most disturbing. We have no satisfactory explanation for this discrepancy.

2. Amplitudes

Critical amplitudes, being generally non-universal quantities, are less widely studied than exponents or critical probabilities, but nevertheless have interesting properties in their own right, particularly when combined in certain universal combinations. There has been little

Table 9. High-density site percolation series expansions $P(p)$ and $S(p)$ in $q = 1 - p$.

	Coefficient of q^n in	
n	$P(p)$	$S(p)$
0	1	1
1	0	0
2	0	4
3	0	20
4	−1	76
5	0	100
6	−4	764
7	−8	−196
8	−23	6480
9	−28	−9316
10	−186	91524
11	48	−240248
12	−1301	1259944
13	1412	−3978772
14	−12292	17210084
15	30384	−59160400
16	−142441	233874228
17	416612	−827672272
18	−1613164	3133896060
19	5161772	−11232481096
20	−18738614	41519884516
21	62391348	−149099004752
22	−219708272	542518525488
23	744185860	−1942522817232
24	−2587326809	6989081250384
25	8835253824	−24903166794592
26	−30515082648	88817460289652
27	104424738432	
28	−358958816131	
29	1227808860812	
30	−4606527025296	

work on this problem for nearly 20 years, since Sykes *et al* [11–13] presented the first extensive study of percolation amplitudes.

Before presenting our own amplitude analysis, we emphasize that it is particularly important to establish a notation when giving critical amplitudes. Sykes *et al* [11–13] defined the amplitudes as follows:

$$P(p) \sim \tilde{B}(p - p_c)^{\beta} \qquad \text{and} \qquad S(p) \sim C'(p - p_c)^{-\gamma'} \qquad \text{for } p > p_c$$

and

$$S(p) \sim C(p_c - p)^{-\gamma} \qquad \text{for } p < p_c.$$

Note that replacing $(p - p_c)$ by $(q_c - q)$ in the above equations leaves the amplitudes unchanged.

In a comprehensive review, Essam [14] redefined the amplitudes in a manner that is more consistent with other critical phenomena amplitudes, such as Ising models, as

$$P(p) \sim B(p/p_c - 1)^{\beta} \qquad \text{and} \qquad S(p) \sim C^{-}(p/p_c - 1)^{-\gamma} \qquad \text{for } p > p_c$$

Table 10. High-density bond percolation series expansions $P(p)$ and $S(p)$ in $q = 1 - p$.

	Coefficient of q^n in	
n	$P(p)$	$S(p)$
0	1	1
1	0	0
2	0	12
3	0	−12
4	0	74
5	0	−104
6	−1	480
7	0	−802
8	−8	3060
9	6	−6964
10	−48	25278
11	66	−62968
12	−279	184996
13	508	−432864
14	−1695	1187324
15	3788	−3076050
16	−11495	9288350
17	28396	−26357140
18	−79820	75320592
19	200686	−198150494
20	−538688	527665840
21	1380702	−1408510098
22	−3703942	3952628584
23	9743716	−11062725766
24	−26142292	30776037860
25	69064132	−83525950010
26	−183971520	227013906708
27	487247714	−619944518810
28	−1299480892	1710417801202
29	3456541714	−4703976303306
30	−9214586926	12899099045260
31	24514580290	−35282516017726
32	−65367671927	96803622888380
33	174407730212	−265803577307442
34	−466154026857	
35	1245497951632	
36	−3328555286658	
37	8895433494310	
38	−23794328607877	
39	63688822876610	

and

$$S(p) \sim C^+(1 - p/p_c)^{-\gamma} \qquad \text{for } p < p_c.$$

He also defined the critical isotherm amplitude E through $P(p_c, h) \sim Eh^{1/\delta}$, where h is a magnetic field, the introduction of which is described in [13]. We expect C^+/C^- to be universal, and also the combination $R' = (C^+)^{1/\delta}E^{-1}B^{1-1/\delta}$. This latter universal quantity follows from lattice–lattice scaling, as developed by Betts *et al* [15], and applied to percolation theory by Stauffer [16]. With our data we have been able to give more accurate estimates of B, C^+ and C^-, and have refined the estimate of E given in [13].

We have estimated the amplitudes in a fairly simple-minded manner, from Padé approximants to $(p_c - p)[S(p)]^{1/\gamma}|_{x=x_c}$ which gives estimates of $p_c[C^+]^{1/\gamma}$, from which C^+ can immediately be obtained. Inhomogeneous differential approximants offer greater accuracy, in principle, but involve substantially greater computation. We found little difference in the achieved accuracy, so we used the computationally simpler Padé method.

Note too that, as the expansion variable for the high-temperature quantities is q, care must be taken to ensure that one calculates the correct amplitude.

In all cases we have used the assumed exact value of the exponents, and the best estimate of the critical probability or the exact value, as available. The results are given in table 11. The error estimates result from the spread of the approximants alone. The exponents are exact, and the site critical probability is very precise. Thus the biasing is not a source of error. The estimates of B and C^+ are quite precise, and lie outside the earlier estimates of Sykes *et al* [11–13]. Our estimates of C^- are far less accurate than the estimates of C^+, but are nevertheless the most accurate estimates yet made of these amplitudes.

From scaling theory we would expect C^+/C^- to be universal, and we find this ratio to be 41^{+31}_{-12} for site percolation, and 49^{+12}_{-9} for bond percolation. While these estimates are consistent with equality, they are not very precise. In order to test the universality of the ratio $R' = (C^-)^{1/\delta}E^{-1}B^{1-1/\delta}$, we have taken the estimates [13] $E = 1.090(25)$ (site) and $E = 1.096(2)$ (bond) which were made assuming the then current best estimate of $\delta = 18$, and performed a simple analysis that shows that a change of δ to the exact value $\delta = 18.2$ produces a corresponding change to $E = 1.100(25)$ (site) and $E = 1.106(2)$. Using our estimate of C^- certainly produces estimates of R' that agree for bond and site percolation, but, because the uncertainty in C^- is so great, this is not a stringent test. Rather, we *assume universality of* C^-/C^+, and instead study the ratio $R = (C^+)^{1/\delta}E^{-1}B^{1-1/\delta}$. Since the error of C^+ is so much smaller, this is a more demanding test. We find the values $R = 1.236(30)$ (site) and $R = 1.238(5)$ (bond), in agreement with the earlier estimate $R \approx 1.25$ given in [14].

The site estimate of E, and hence R, could probably be improved upon slightly by re-analysing the original data [13] using the best Monte Carlo estimate [10] of p_c, but we have not done this.

The amplitudes C^+ and B quoted in [14] are, in fact, the values obtained by Sykes *et al.* Thus the entries in table 3 of [14] labelled C^+ and B should have been labelled C and $\tilde{B}$.

3. Summary

We have investigated both bond and site animals, enumerated by both perimeter and area. We have obtained the asymptotic form for the number of bond and site animals of a given area, the mean perimeter and mean-square perimeter and the mean area and mean-square area. The mean perimeter, for both bond and site animals is found to behave qualitatively like the corresponding quantity for directed animals and self-avoiding polygons, in that $\langle p\rangle_n \sim n$ and $\langle p^2\rangle_n \sim n^2$. Similarly, the mean area for both bond and site animals appears to be qualitatively similar to self-avoiding polygons, in that $\langle a\rangle_n \sim n^{\frac{3}{2}}$ and $\langle a^2\rangle_n \sim n^3$. The variance of all these quantities has also been calculated.

For percolation, we give both biased and unbiased estimates of the bond and site critical probabilities, the high- and low-temperature mean-size exponents and the percolation probability exponent. Apart from some unexplained results for the high-temperature bond percolation mean-size exponent, all our results are consistent with the presumably exact exponent estimates from conformal invariance theory [17], notably $\beta = \frac{5}{36}$ and $\gamma = \frac{43}{18}$; the exact value $p_c(\text{bond}) = \frac{1}{2}$ and the best Monte Carlo estimate [10] for $p_c(\text{site})$. Critical amplitudes for all the above quantities are also estimated.

In table 11 we summarize all our results.

Table 11. Summary of results.

Description	Quantity	Site	Bond
Number of animals	(n = area) c_n	$(0.3160(5))(4.06265(5))^n/n$	$(0.5084(3))(5.2082(14))^n/n$
Mean perimeter	(n = area) $\langle p\rangle_n$	$1.195n$	$1.634n$
variance of perimeter	$\mathrm{var}(\langle p\rangle_n)$	$0.3885n$	$0.48n$
Mean-square perimeter	(n = area) $\langle p^2\rangle_n$	1.195^2n^2	1.634^2n^2
Mean area	(n = perimeter) $\langle a\rangle_n$	$0.30n^{1.5}$	$0.292n^{1.5}$
Mean-square area	(n = perimeter) $\langle a^2\rangle_n$	0.30^2n^3	0.292^2n^3
Variance of area	$\mathrm{var}(\langle a\rangle_n)$	$0.0163\sqrt{n}$	$0.014\sqrt{n}$
Critical probability[a]	p_c	0.5928(2)	0.5005(10)
Critical probability[b]	p_c	0.5927(1)	0.50000(8)
Mean size exponent[a]	γ	2.40(4)	2.42(4)
Mean size exponent[b]	γ	2.392(7)	2.389(1)
Critical amplitude[b]	C^+	0.5745(30)	0.785(6)
Mean size exponent[a]	γ'	2.0(6)	2.04(8)
Mean size exponent[b]	γ'	2.0(4)	2.06(2)
Critical amplitude[b]	C^-	0.014(6)	0.016(3)
Perc. probability exponent[b]	β	0.136(9)	0.10(8)
Perc. probability exponent[b]	β	0.1383(10)	0.1387(3)
Critical amplitude[b]	B	1.4290(3)	1.4139(15)

[a] Unbiased.
[b] Biased.

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References

[1] Conway A R 1995 *J. Phys. A: Math. Gen.* **28** 335–49
[2] Redelmeier D H 1981 *Disc. Math.* **36** 191–203
[3] Guttmann A J 1982 *J. Phys. A: Math. Gen.* **15** 1987–90
[4] Conway A R 1994 *J. Phys. A: Math. Gen.* **21** 7007
[5] Enting I G and Guttmann A J 1990 *J. Stat. Phys.* **58** 475–84
[6] Enting I G and Guttmann A J 1992 *J. Phys. A: Math. Gen.* **25** 2791–807
[7] Cardy J L and Guttmann A J 1993 *J. Phys. A: Math. Gen.* **26** 2485–94
[8] Adler J 1994 *Comput. Phys.* **8** 287–95
[9] Roskies R Z 1981 *Phys. Rev.* B **24** 5305
[10] Ziff R M 1992 *Phys. Rev. Lett.* **69** 2670
[11] Sykes M F, Gaunt D S and Glen M 1976 *J. Phys. A: Math. Gen.* **9** 725–30
[12] Sykes M F, Gaunt D S and Glen M 1976 *J. Phys. A: Math. Gen.* **9** 97–103

[13] Gaunt D S and Sykes M F 1976 *J. Phys. A: Math. Gen.* 9 1109–16
[14] Essam J W 1980 *Rep. Prog. Phys.* 43 833–912
[15] Betts D D, Guttmann A J and Joyce G S 1971 *J. Phys. C: Solid State Phys.* 4 1994
[16] Stauffer D 1976 *Z. Phys.* B **25** 351–9
[17] Friedan D, Qui Z and Shenker S 1984 *Phys. Rev. Lett.* **52** 1875